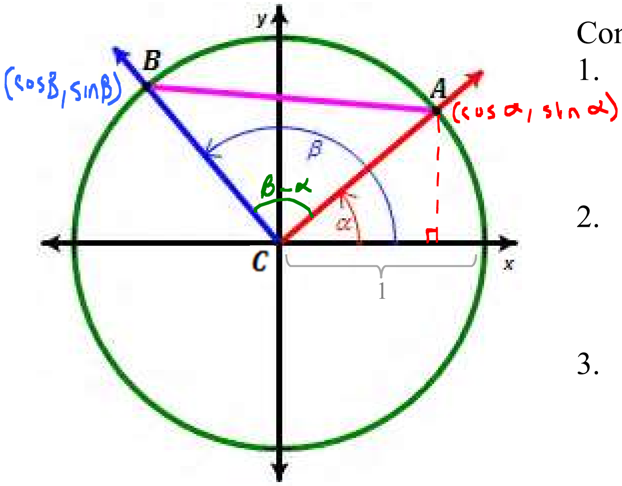
Sec 4.2 – Trigonometric Identities

Sum & Difference Identities

Name:



Consider the diagram at the right of a unit circle.

1. First, determine the coordinates of point A in terms of α.

   $A: (\cos\alpha, \sin\alpha)$

2. First, determine the coordinates of point B in terms of β.

   $B: (\cos\beta, \sin\beta)$

3. Using those coordinates and the distance formula, find the distance between AB in terms of α and β.

   $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

   $AB = \sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2}$

   $AB = \sqrt{\cos^2\beta - 2\cos\beta\cos\alpha + \cos^2\alpha + \sin^2\beta - 2\sin\beta\sin\alpha + \sin^2\alpha}$

   $AB = \sqrt{(\cos^2\beta + \sin^2\beta) + (\cos^2\alpha + \sin^2\alpha) - 2\sin\beta\sin\alpha - 2\cos\beta\cos\alpha}$

   $AB = \sqrt{1 + 1 - 2(\sin\beta\sin\alpha + \cos\beta\cos\alpha)}$

4. Using the Law of Cosines and triangle ABC, find the length of side AB in terms of α and β.

   $c = \sqrt{a^2 + b^2 - 2ab\cos C}$

   $AB = \sqrt{1^2 + 1^2 - 2(1)(1)\cos(\beta - \alpha)}$

   $AB = \sqrt{2 - 2\cos(\beta - \alpha)}$

5. Use the two unique descriptions of the length of AB create a new trigonometric identitity.

   $\sqrt{2 - 2\cos(\beta-\alpha)} = \sqrt{1 + 1 - 2(\sin\beta\sin\alpha + \cos\alpha\cos\beta)}$

   $2 - 2\cos(\beta-\alpha) = 2 - 2(\sin\beta\sin\alpha + \cos\alpha\cos\beta)$

   $2 - 2\cos(\beta-\alpha) = 2 - 2(\sin\beta\sin\alpha + \cos\alpha\cos\beta)$

   $-2 \qquad\qquad\qquad -2$

   $\dfrac{-2\cos(\beta-\alpha)}{-2} = \dfrac{-2(\sin\beta\sin\alpha + \cos\beta\cos\alpha)}{-2}$

   $\cos(\beta-\alpha) = \sin\beta\sin\alpha + \cos\beta\cos\alpha$

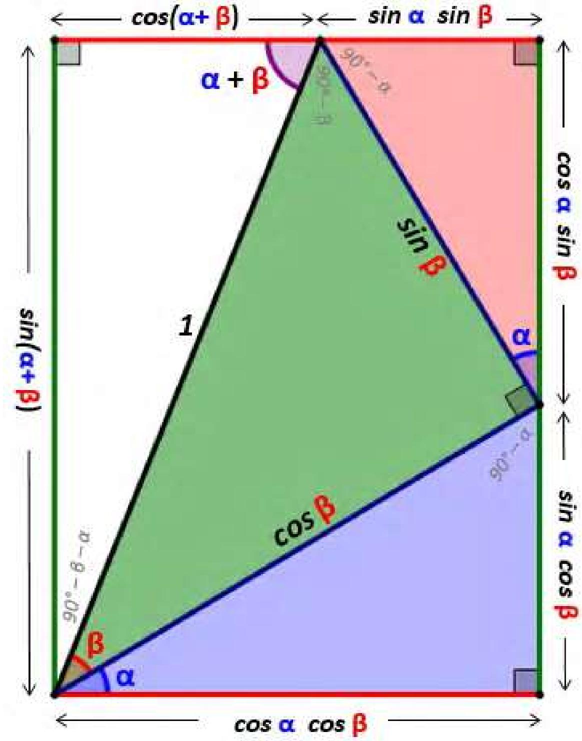
   — OR —

   $\cos(\beta-\alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$

   — OR —

   $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

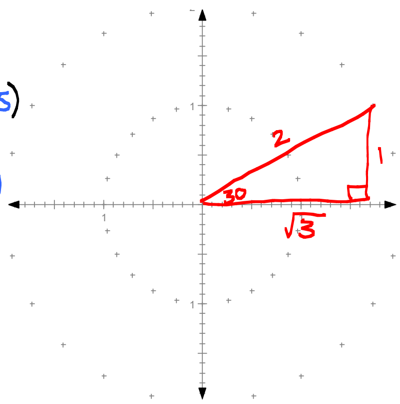
Alternately, you could use the following elegant diagram to show the sum identities hold true for angles from 0° to 90° or 0 to $\frac{\pi}{2}$ by realizing opposite sides of a rectangle must have the same measure. Further, the difference identities can be determined by replacing β with negative β and simplifying.

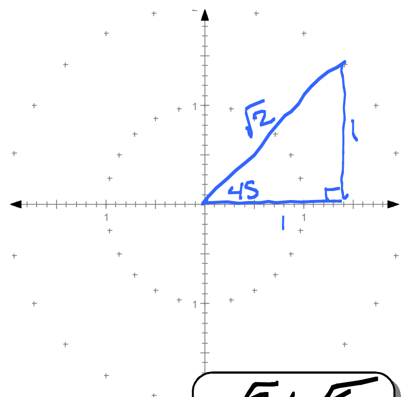


$$sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

1. Find the **exact** value of $\sin(75°)$
(using sum and difference identities)
$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$$\sin(30+45) = \sin(30)\cos(45) + \cos(30)\sin(45)$$
$$= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{1+\sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$$

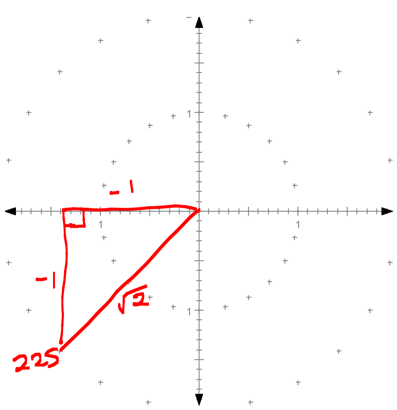


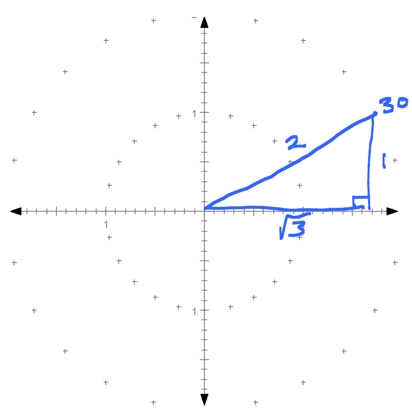


$$\boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$$

2. Find the **exact** value of $\cos(255°)$
(using sum and difference identities)
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$$\cos(225+30) = \cos(225)\cos(30) - \sin(225)\sin(30)$$
$$= \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$
$$= \frac{-\sqrt{3}}{2\sqrt{2}} - \frac{-1}{2\sqrt{2}}$$
$$= \frac{-\sqrt{3}+1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{-\sqrt{6}+\sqrt{2}}{4}}$$





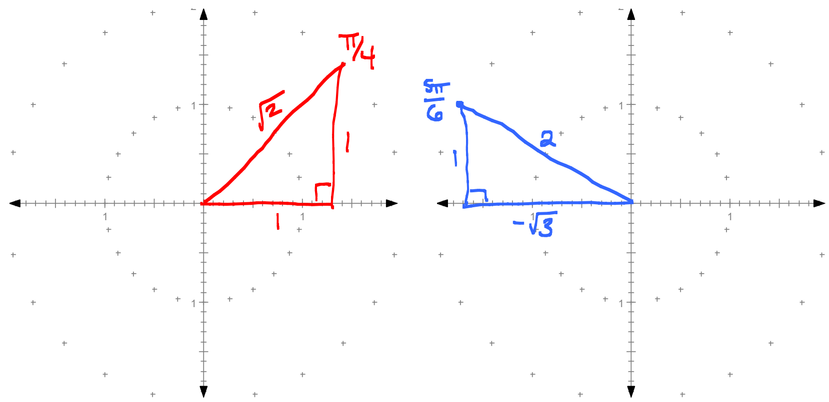
$$\boxed{\frac{-\sqrt{6}+\sqrt{2}}{4}}$$

3. Find the **exact** value of $\sin\left(\frac{13\pi}{12}\right)$

(using sum and difference identities)

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$$\frac{3\pi}{12} + \frac{10\pi}{12} = \frac{\pi}{4} + \frac{5\pi}{6} = \frac{13\pi}{12}$$



$$\sin\left(\frac{\pi}{4}+\frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{5\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{6}\right)$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$= \frac{-\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{-\sqrt{3}+1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{-\sqrt{6}+\sqrt{2}}{4}}$$

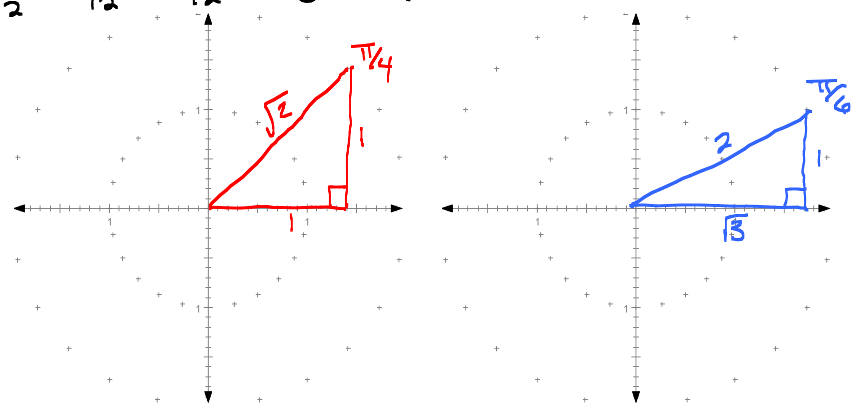
$$\boxed{\frac{-\sqrt{6}+\sqrt{2}}{4}}$$

4. Find the **exact** value of $\cos\left(\frac{5\pi}{12}\right)$

(using sum and difference identities)

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$$\frac{5\pi}{12} = \frac{2\pi}{12} + \frac{3\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$$



$$\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

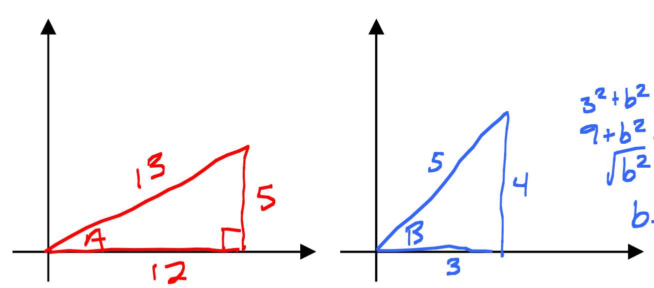
$$\boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

5. Given that $\sin(A) = \frac{5}{13}$ and $\cos(B) = \frac{3}{5}$. Also, assume A & B are in the first quadrant.

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

Find the **exact** value of $\sin(A - B) =$

$\sin(A) = \frac{5 \leftarrow \text{OPP}}{13 \leftarrow \text{HYP}}$ $\qquad$ $\cos(B) = \frac{3 \leftarrow \text{ADJ}}{5 \leftarrow \text{HYP}}$



$3^2 + b^2 = 5^2$
$9 + b^2 = 25$
$\sqrt{b^2} = \sqrt{16}$
$b = 4$

$a^2 + 5^2 = 13^2$
$a^2 + 25 = 169$
$\quad -25 \quad -25$
$\sqrt{a^2} = \sqrt{144}$
$a = 12$

$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

$$= \left(\frac{5}{13}\right)\left(\frac{3}{5}\right) - \left(\frac{12}{13}\right)\left(\frac{4}{5}\right)$$

$$= \frac{15}{65} - \frac{48}{65}$$

$$= \boxed{\frac{33}{65}}$$

$$\boxed{\frac{33}{65}}$$

6. Given that $\cos(A)=\dfrac{9}{13}$ and $\cos(B)=\dfrac{1}{5}$. Also, assume A & B are in the first quadrant.

$\cos(\alpha \pm \beta)=\cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

Find the **exact** value of $\cos(A+B)=$

$$\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$$

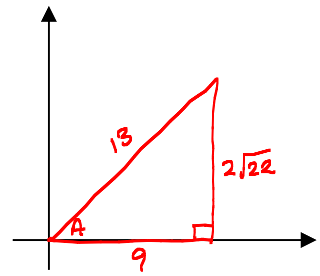
$$=\left(\frac{9}{13}\right)\left(\frac{1}{5}\right)-\left(\frac{2\sqrt{22}}{13}\right)\left(\frac{2\sqrt{6}}{5}\right)$$

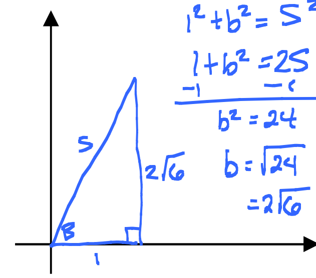
$$=\frac{9}{65}-\frac{4\sqrt{132}}{65}=\frac{9}{65}-\frac{4\cdot 2\sqrt{33}}{65}$$

$$=\frac{9}{65}-\frac{8\sqrt{33}}{65}=\boxed{\frac{9-8\sqrt{33}}{65}}$$

$\cos A = \frac{9}{13}$ ← ADJ, ← HYP

$\cos B = \frac{1}{5}$ ← ADJ, ← HYP





$$9^2+b^2=13^2$$
$$81+b^2=169$$
$$-81 \quad -81$$
$$\sqrt{b^2}=\sqrt{88}=2\sqrt{22}$$

$$1^2+b^2=5^2$$
$$1+b^2=25$$
$$-1 \quad -1$$
$$b^2=24$$
$$b=\sqrt{24}=2\sqrt{6}$$

$$\boxed{\frac{9-8\sqrt{33}}{65}}$$

Simplify the following trigonometric expressions using the sum and difference identities.

7. $\sin(\pi+\theta)=\sin(\pi)\cos(\theta)+\cos(\pi)\sin(\theta)$

$$=\underbrace{(0)\cdot\cos\theta}+\underbrace{(-1)\cdot\sin\theta}$$

$$=0-\sin\theta$$

$$=\boxed{-\sin\theta}$$

8. $\underbrace{\cos(\theta+\theta)}=\underbrace{\cos(\theta)\cos(\theta)}-\underbrace{\sin(\theta)\sin(\theta)}$

$$\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$$

Determine the sum identity for tangent using the sum identities for sine and cosine.

$$\mathrm{Tan}(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\,\mathrm{si}} \cdot \frac{\frac{1}{\cos\alpha\cos\beta}}{\frac{1}{\cos\alpha\cos\beta}}$$

$$\frac{\frac{1}{\cos\alpha\cos\beta}}{\frac{1}{\cos\alpha\cos\beta}} = 1$$

$$= \frac{\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} + \frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} - \frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}$$

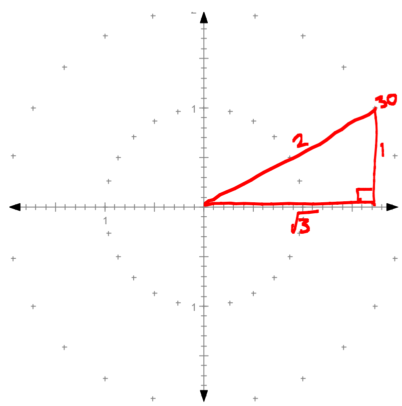
$$= \frac{\frac{\sin(\alpha)}{\cos(\alpha)}\frac{\cancel{\cos(\beta)}}{\cancel{\cos(\beta)}} + \frac{\cancel{\cos(\alpha)}}{\cancel{\cos(\alpha)}}\frac{\sin(\beta)}{\cos(\beta)}}{\frac{\cancel{\cos(\alpha)}}{\cancel{\cos(\alpha)}}\frac{\cancel{\cos(\beta)}}{\cancel{\cos(\beta)}} - \frac{\sin(\alpha)}{\cos(\alpha)}\frac{\sin(\beta)}{\cos(\beta)}} = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$$

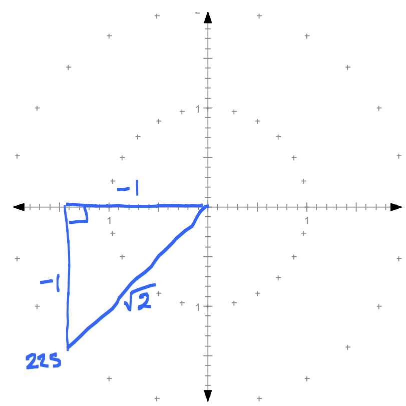
9. Find the **exact** value of $\tan(255°)$

(using sum and difference identities)

$$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

$255 = 30 + 225$





$$\tan(30+225) = \frac{\tan(30)+\tan(225)}{1-\tan(30)\tan(225)}$$

$$= \frac{\left(\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{1}\right)}{1-\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{1}\right)}$$

$$= \boxed{\frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}}} \text{ OR } \frac{\frac{\sqrt{3}}{3}+\frac{3}{3}}{\frac{3}{3}-\frac{\sqrt{3}}{3}} = \frac{\sqrt{3}+3}{3} \div \frac{3-\sqrt{3}}{3} = \boxed{\frac{\sqrt{3}+3}{3-\sqrt{3}}} \cdot \frac{(3+\sqrt{3})}{(3+\sqrt{3})} = \frac{3\sqrt{3}+3+9+3\sqrt{3}}{9+3\sqrt{3}-3\sqrt{3}-3}$$

OK

BETTER

$$= \frac{6\sqrt{3}+12}{6} = \boxed{\sqrt{3}+2}$$

BEST

10. Given that $\sin(A)=\frac{5}{13}$ and $\cos(B)=\frac{8}{17}$. Also, assume A & B are in the first quadrant.

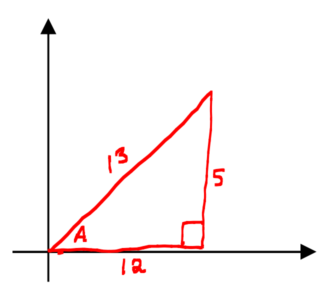
$$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

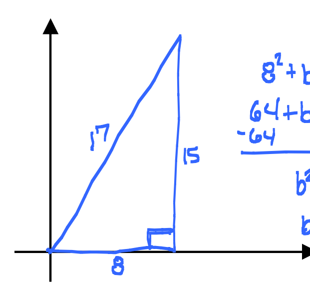
Find the **exact** value of $\tan(A-B)=$

$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

$$= \frac{\frac{2\cdot 5}{2\cdot 12} - \frac{15\cdot 3}{8\cdot 3}}{1+\left(\frac{5}{12}\right)\left(\frac{15}{8}\right)} = \frac{\frac{10}{24}-\frac{45}{24}}{1+\cancel{\frac{75}{96}}\,\frac{25}{32}}$$

$$= \frac{\frac{-35}{24}}{\frac{32}{32}+\frac{25}{32}} = \frac{\frac{-35}{24}}{\frac{57}{32}} = \frac{-35}{24}\cdot\frac{32}{57} = \boxed{\frac{-140}{171}}$$





$a^2+5^2=13^2$

$a^2+25=169$
$-25 \quad -25$

$\sqrt{a^2} = \sqrt{144}$

$a = 12$

$8^2+b^2=17^2$

$64+b^2=289$
$-64 \quad -64$

$b^2=225$

$b=15$

$$\boxed{\frac{-140}{171}}$$