Sec 4.3 – Trigonometric Identities

Half Angle Identities

Name:

Starting with the double angle identities, create the half angle identities:

Double-Angle Identities

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\cos(2\theta) = 1 - 2\sin^2\theta$
- $\cos(2\theta) = 2\cos^2\theta - 1$
- $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

1. Let $\theta = \frac{\alpha}{2}$ using the identity $\cos(2\theta) = 1 - 2\sin^2\theta$ and solve for $\sin\theta$

$$\cos\left(2\,\frac{\alpha}{2}\right) = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$$

$$\cos(\alpha) = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$$

$$-1 \qquad -1$$

$$\frac{\cos(\alpha) - 1}{-2} = \frac{-2\sin^2\left(\frac{\alpha}{2}\right)}{-2}$$

$$\pm\sqrt{\frac{-\cos(\alpha)+1}{2}} = \sqrt{\sin^2\left(\frac{\alpha}{2}\right)}$$

$$\pm\sqrt{\frac{1-\cos(\alpha)}{2}} = \sin\left(\frac{\alpha}{2}\right)$$

2. Let $\theta = \frac{\alpha}{2}$ using the identity $\cos(2\theta) = 2\cos^2\theta - 1$ and solve for $\cos\theta$

$$\cos\left(2\,\frac{\alpha}{2}\right) = 2\cos^2\left(\frac{\alpha}{2}\right) - 1$$

$$\cos(\alpha) = 2\cos^2\left(\frac{\alpha}{2}\right) - 1$$

$$+1 \qquad +1$$

$$\frac{\cos(\alpha) + 1}{2} = \frac{2\cos^2\left(\frac{\alpha}{2}\right)}{2}$$

$$\pm\sqrt{\frac{\cos(\alpha)+1}{2}} = \sqrt{\cos^2\left(\frac{\alpha}{2}\right)}$$

$$\pm\sqrt{\frac{1+\cos(\alpha)}{2}} = \cos\left(\frac{\alpha}{2}\right)$$

3. Use the two identities you just created in problems 1 and 2, simplify the following to create the tangent half angle identity.

$$\tan\frac{\theta}{2} = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \frac{\pm\sqrt{\frac{1-\cos(\alpha)}{2}}}{\pm\sqrt{\frac{1+\cos(\alpha)}{2}}} = \pm\sqrt{\frac{\frac{1-\cos(\alpha)}{2}}{\frac{1+\cos(\alpha)}{2}}} = \pm\sqrt{\frac{1-\cos(\alpha)}{\cancel{2}} \cdot \frac{\cancel{2}}{1+\cos(\alpha)}}$$

$$= \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}}$$

Half-Angle Identities

- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$
- $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

4. Find the **exact** value of $\cos(112.5°)$
(using half angle identities)

LET $\theta = 225°$

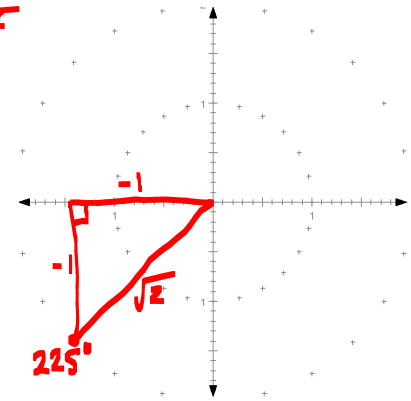
SINCE 112.5 EXISTS IN QUAD.II
COSINE WILL BE NEGATIVE

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\left(\frac{225}{2}\right) = -\sqrt{\frac{1+\cos 225}{2}} = -\sqrt{\frac{1+\left(-1/\sqrt{2}\right)}{2}} = -\sqrt{\frac{1-\sqrt{2}/2}{2}}$$

$$= -\sqrt{\frac{\frac{2}{2}-\frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{2}}{4}} = \frac{-\sqrt{2-\sqrt{2}}}{2}$$

$$\boxed{\frac{-\sqrt{2-\sqrt{2}}}{2}}$$



---

5. Find the **exact** value of $\sin\left(\frac{\pi}{12}\right)$
(using half angle identities)

LET $\theta = \pi/6$

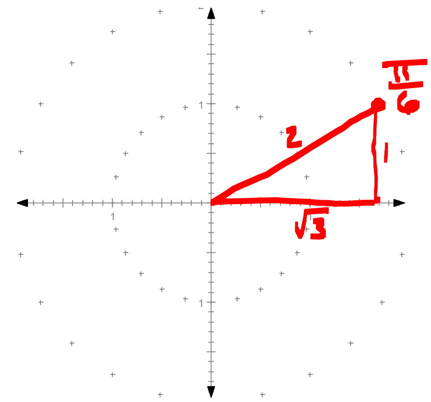
SINCE $\pi/12$ IS IN QUAD I
SINE WILL BE POSITIVE

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\left(\frac{\pi/6}{2}\right) = \sqrt{\frac{1-\cos\pi/6}{2}} = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \sqrt{\frac{2/2-\sqrt{3}/2}{2}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{2}\cdot\frac{1}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

$$\boxed{\frac{\sqrt{2-\sqrt{3}}}{2}}$$



---

6. Find the **exact** value of $\tan(15°)$
(using half angle identities)

LET $\theta = 30$

SINCE 15° EXISTS IN QUAD I
TANGENT WILL BE POSITIVE.

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$$

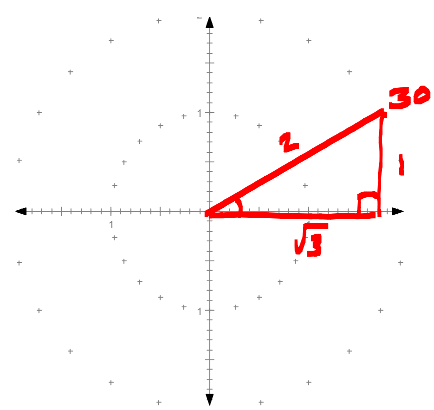
$$\tan\left(\frac{30}{2}\right) = \sqrt{\frac{1-\cos(30)}{1+\cos(30)}} = \sqrt{\frac{1-\sqrt{3}/2}{1+\sqrt{3}/2}} = \sqrt{\frac{\frac{2}{2}-\frac{\sqrt{3}}{2}}{\frac{2}{2}+\frac{\sqrt{3}}{2}}}$$

$$= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{\frac{2+\sqrt{3}}{2}}} = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}} \quad \frac{(2-\sqrt{3})}{(2-\sqrt{3})}$$

$$\boxed{\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}}$$

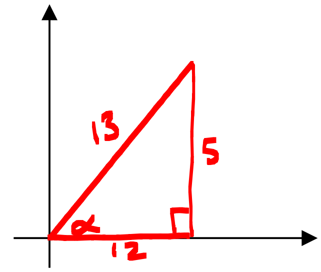
$$= \sqrt{\frac{4-4\sqrt{3}+3}{4-3}} = \sqrt{\frac{7-4\sqrt{3}}{1}} = \sqrt{7-4\sqrt{3}}$$

BETTER



7. Given that $\sin(\alpha)=\frac{5}{13}$ and assume angle α is in the first quadrant.

IN QUAD I SINE IS POSITIVE



$a^2+5^2=13^2$

$a=12$

a. Find the **exact** value of $\sin\left(\frac{\alpha}{2}\right)= \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-12/13}{2}}$

$\boxed{\frac{\sqrt{26}}{26}}$ $= \sqrt{\frac{13/13-12/13}{2}} = \sqrt{\frac{1/13}{2}} = \sqrt{\frac{1}{26}} = \frac{1}{\sqrt{26}} = \frac{\sqrt{26}}{26}$

IN QUAD I COSINE IS POSITIVE

b. Find the **exact** value of $\cos\left(\frac{\alpha}{2}\right)= \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1+12/13}{2}}$

$\boxed{\frac{5\sqrt{26}}{26}}$ $= \sqrt{\frac{13/13+12/13}{2}} = \sqrt{\frac{25/13}{2}} = \sqrt{\frac{25}{26}} = \frac{5}{\sqrt{26}} = \frac{5\sqrt{26}}{26}$

c. Find the **exact** value of $\tan\left(\frac{\alpha}{2}\right)= \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \sqrt{\frac{1-12/13}{1+12/13}} = \sqrt{\frac{13/13-12/13}{13/13+12/13}}$

$\boxed{\frac{1}{5}}$ $= \sqrt{\frac{\frac{1}{13}}{\frac{25}{13}}} = \sqrt{\frac{1}{13}\cdot\frac{13}{25}} = \sqrt{\frac{1}{25}} = \frac{1}{5}$

8. Verify the following identities using the half angle identities.

a. $2-2\cos^2\left(\frac{\theta}{2}\right)=1-\cos\theta$

$2-2\left(\pm\sqrt{\frac{1+\cos\theta}{2}}\right)^2 = 1-\cos\theta$: REPLACE $\cos\left(\frac{\theta}{2}\right)$ USING HALF ANGLE IDENTITY

$2-2\left(\frac{1+\cos\theta}{2}\right) = 1-\cos\theta$: THE SQUARED AND SQUARE ROOT ARE INVERSE OPERATIONS

$2-\cancel{2}\left(\frac{1+\cos\theta}{\cancel{2}}\right) = 1-\cos\theta$: SIMPLIFY

$2-(1+\cos\theta) = 1-\cos\theta$: SIMPLIFY

$2-1-\cos\theta = 1-\cos\theta$: SIMPLIFY

$1-\cos\theta = 1-\cos\theta$ ✓

b. $\frac{\sin\theta}{1+\cos\theta}=\tan\frac{\theta}{2}$: REPLACE $\tan\left(\frac{\theta}{2}\right)$ WITH HALF ANGLE IDENTITY

$\frac{\sin\theta}{1+\cos\theta} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{(1+\cos\theta)}{(1+\cos\theta)}}$: MULTIPLY BY $\frac{(1+\cos\theta)}{(1+\cos\theta)}=1$

$\frac{\sin\theta}{1+\cos\theta} = \sqrt{\frac{1-\cos\theta+\cos\theta-\cos^2\theta}{(1+\cos\theta)^2}}$: EXPAND THE NUMERATOR

$\frac{\sin\theta}{1+\cos\theta} = \sqrt{\frac{1-\cos^2\theta}{(1+\cos\theta)^2}}$: SIMPLIFY

$\frac{\sin\theta}{1+\cos\theta} = \sqrt{\frac{\sin^2\theta}{(1+\cos\theta)^2}}$: REPLACE $1-\cos^2\theta$ WITH $\sin^2\theta$

$\frac{\sin\theta}{1+\cos\theta} = \frac{\sqrt{\sin^2\theta}}{\sqrt{(1+\cos\theta)^2}}$: REWRITE RADICAL

$\frac{\sin\theta}{1+\cos\theta} = \frac{\sin\theta}{1+\cos\theta}$: INVERSE OPERATIONS